

# Partial translation of an article by Paul Drude in 1904

Translation by A. J. Sederberg\*

Discussion by B. H. McGuyer†

*Department of Physics, Princeton University, Princeton, New Jersey 08544, USA*

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\*Present address: Division of Biological Sciences, University of Chicago, 5812 South Ellis Avenue, Chicago, IL 60637, USA.

†Present address: Department of Physics, Columbia University, 538 West 120th Street, New York, NY 10027-5255, USA.

# 1 Introduction

The German physicist Paul Drude (1863–1906) [1] published a few articles [2, 3, 4, 5] on the physics of Tesla transformers (or Tesla coils) in the beginning of the 20th century, during the era of wireless telegraphy (or early radio). These articles are of historical interest to the modeling of solenoids and Tesla transformers. In particular, Drude’s 1904 article [4] is still cited as an important reference to justify the conventional equivalent circuit (or lumped-element model) for a Tesla transformer [6, 7]. Unfortunately, no official English translation of this article exists. A partial translation covering pp. 512–514 & 560–561 is available online [8].

This document presents an unofficial, partial translation of Drude [4], covering the derivation of an equivalent circuit for a half-wave Tesla transformer (pp. 512–519), as well as the summary of results at its end (pp. 560–561). The equation numbering has been kept the same, and approximate page markings are included for reference. The remaining, untranslated pages (pp. 520–560) analyze various wireless telegraphy applications. For reference, discussions on modeling Tesla transformers are available in [6, 7, 9, 10, 11, 12, 13].

During this translation, we were surprised to find Drude’s prediction that *the mutual inductance in the equivalent circuit for a Tesla transformer should be nonreciprocal* ( $M_{12} \neq M_{21}$ )! This mostly forgotten prediction is discussed in a multi-edition book on wireless telegraphy by J. A. Fleming [11], and is mentioned in the books on inductance coils by E. T. Jones [14, 15]. It is likely mentioned in other books from the wireless-telegraphy era, where Tesla transformers may be called “oscillation transformers” or “Thomson coils” (after Elihu Thomson). Relatedly, Hund [16] treats mutual inductances in general as nonreciprocal, though does not mention Drude [4].

This document ends with a discussion of the derivation of the equivalent circuit in Drude [4]. Some errors seem to have prevented the completion of this derivation, and lead to a different equivalent inductance for a resonant solenoid than that of Drude’s 1902 article [2]. We present a revised derivation which resolves this disagreement.

# 2 Bibliographic information

**Author:** Paul Karl Ludwig Drude

**Title:** “Über induktive Erregung zweier elektrischer Schwingungskreise mit Anwendung auf Perioden- und Dämpfungsmessung, Teslatransformatoren und drahtlose Telegraphie”

**Translated Title:** “Of inductive excitation of two electric resonant circuits with application to measurement of oscillation periods and damping, Tesla coils, and wireless telegraphy”

**Journal:** Annalen der Physik (abbreviated “Ann. Phys.” or “Ann. d. Phys.”)

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### 3 Partial translation

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Start of article and translation

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#### 5. Of inductive excitation of two electric resonant circuits with application to measurement of oscillation periods and damping, Tesla coils, and wireless telegraphy; by P. Drude.

Contents: Introduction. I. Definition and integration of the differential equations p. 513. II. The magnetic coupling is very small p. 521. III. Measuring the period and the damping p. 525. 1. The maximum amplitude p. 528. 2. The integral effect p. 530. IV. The magnetic coupling is not very small p. 534. V. The effectiveness of the Tesla transformer p. 540. VI. Dependence of the Tesla effect on damping and coupling p. 544. VII. Application to wireless telegraphy p. 550. a) Simple or loosely-coupled receiver p. 551. b) Tightly-coupled receiver p. 554. Main results p. 560.

J. v. Geitler<sup>1</sup>, B. Galitzin<sup>2</sup>, A. Oberbeck<sup>3</sup> and Domalip and Koláček<sup>4</sup> have proven that, if two electric oscillating circuits interact strongly enough, each no longer has only one, but two, natural frequencies; this remains true even when the two systems are attuned with each other, that is to say, if they share a natural frequency and have weak or no interaction. This problem was later handled by M. Wien<sup>5</sup> in a general and complete manner. Wien<sup>6</sup> also applies his results to wireless telegraphy after the Braun System. V. Bjerknes<sup>7</sup> worked out in detail the case of very weak coupling, considering oscillation frequency and damping measurements through reference to the so-called resonance curve.

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The treatment of this problem here differs from the aforementioned work in the following ways:

1. The solution of the differential equations, especially the calculations of the amplitudes from the initial conditions, will be presented in a mathematically transparent way such that miscalculations are easily avoided, and which remains valid for more complicated applications, such as a strongly coupled transmitter and receiver in wireless telegraphy.<sup>8</sup>
2. The induced circuit will not, as in the cited work, be treated simply as a constant current-carrying wire, with a capacitor on the end, but rather in accordance with the actual conditions.
3. Contrary to Wien's work, a difference appears in the results regarding the damping of both natural frequencies in a strongly coupled system.<sup>9</sup>
4. Bjerknes only discusses the resonance curve of integral effects in detail. Here we shall also consult the resonance curve for maximum amplitude and thereby propose a simple experimental method for the determination of the individual attenuation of both oscillating circuits.
5. The question of how best to construct a Tesla coil will be further addressed. Its resolution requires further experimental study still.

#### I. Definition and Integration of the Differential Equations

We initially assume that the secondary coil (e.g., a Tesla coil) lies centered and symmetric to the primary circuit. The general results also apply to any orientation of the secondary coil. The primary circuit includes

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<sup>1</sup>J. v. Geitler, Sitzungsber. d. k. Akad. d. Wissensch. zu Wien, Februar u. Oktober 1895.

<sup>2</sup>Fürst B. Galitzin, Petersb. Ber., Mai u. Juni 1895.

<sup>3</sup>A. Oberbeck, Wied. Ann. **55**, p. 623. 1895.

<sup>4</sup>R. Domalip u. F. Koláček, Wied. Ann. **57**, p. 731. 1896.

<sup>5</sup>M. Wien, Wied. Ann. **61**, p. 151. 1897.

<sup>6</sup>M. Wien, Ann. d. Phys. **8**, p. 686. 1902.

<sup>7</sup>V. Bjerknes, Wied. Ann. **55**, p. 120. 1895.

<sup>8</sup>Addendum: I see that even Domalip and Koláček have chosen a very similar treatment to this.

<sup>9</sup>It follows for the constants in the differential equations on p. 518 the conclusion  $L_{12} < L_{21}$ , whereas  $L_{12} = L_{21}$  was otherwise assumed.

the (electromagnetically measured) capacitance  $C_1$ . Let the potential difference between the plates of the capacitor  $C_1$  at time  $t$  be  $V_1$ , and let the current strength, that we can assume constant throughout the

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entire primary circuit (since  $C_1$  is chosen to be very large), be  $i_1$ . Let the number of magnetic field lines, which at any time loop around the primary circuit, be  $N_1$ . Then it follows:

$$i_1 = -C_1 \frac{dV_1}{dt}, \quad (1)$$

$$\frac{d^2 N_1}{dt^2} + w_1 \frac{di_1}{dt} + \frac{i_1}{C_1} = 0. \quad (2)$$

$w_1$  is a coefficient on which the damping of the primary circuit depends (resistance from the wire and sparks, as well as the [minor] loss to radiation of the capacitor, eventually also the absorption of electrons in its dielectric).  $w_1$  will also be assumed constant in time. Should this assumption not be met, then during the oscillation  $w_1$  may be understood as the average.  $N_1$  depends on  $i_1$  and the current strength  $i_2$  in the second coil, which is not constant along the coil.

If we choose the axis of the coil (as if wrapped around a cylinder) to be the  $z$ -direction,  $z = 0$  lies in the middle of the coil, while both ends of the coil lie at  $z = \pm h/2$  (so that  $h$  is the height of the coil), then we can write for  $i_2$  the first element of the Fourier series (fundamental mode):

$$i_2 = i_2^0 \cos\left(\frac{\pi z}{h}\right). \quad (3)$$

$i_2^0$  is the current strength in the middle of the coil. In doing this so it is implied that the coil ends are free, without applied capacitance, so that it must be that  $i_2 = 0$  for  $z = \pm h/2$ .

The field lines  $N_1$  now fall into two parts:

$$N_1 = N_{11} + N_{12}, \quad (4)$$

of which the first part shall denote the field lines which only enclose the primary circuit, while  $N_{12}$  shall count the field lines which go around the primary circuit as well as the inductor coil. (See Fig. 1.)

For all the  $N_{11}$  field lines the magnetomotive strength is the same, namely  $4\pi i_1$ , if circuit 1 can be considered linearly, as we wish to assume and as is adequately seen in practice. Then  $W_{11}$  is the magnetic

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resistance of the (connected in parallel) tube strength  $N_{11}$ , which is:<sup>10</sup>

$$N_{11} = \frac{4\pi i_1}{W_{11}}. \quad (5)$$

For the field lines  $N_{12}$ , the magnetomotive strength is not uniform, since the coil cannot be treated as a linear circuit. If one of these field lines cuts through the coil plane at position  $z = \pm a$ , then the magnetic force for this line is:<sup>11</sup>

$$4\pi i_1 + 4\pi \frac{n}{h} \int_{-a}^{+a} i_2 dz = 4\pi i_1 + 8n i_2^0 \sin\left(\frac{\pi a}{h}\right),$$

if  $n$  denotes the total number of coil windings, so that along segment  $dz$ , there are  $(n/h)dz$  windings. We can write an identical average electromotive force for the various field lines  $N_{12}$ :

$$4\pi i_1 + 8n i_2^0 \sin\left(\pi \frac{a_1}{h}\right),$$

in which  $a_1$  is smaller than  $h/2$  and larger than  $a'$ , where the coil shell of the shortest field line that just barely loops around  $i_1$  may be sliced.

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<sup>10</sup>See for example P. Drude, Physik d. Äthers p. 72. Stuttgart 1894.

<sup>11</sup>The + sign before the second term on the right hand side applies if  $i_1$  and  $i_2$  are positive, which is expected in the same direction.

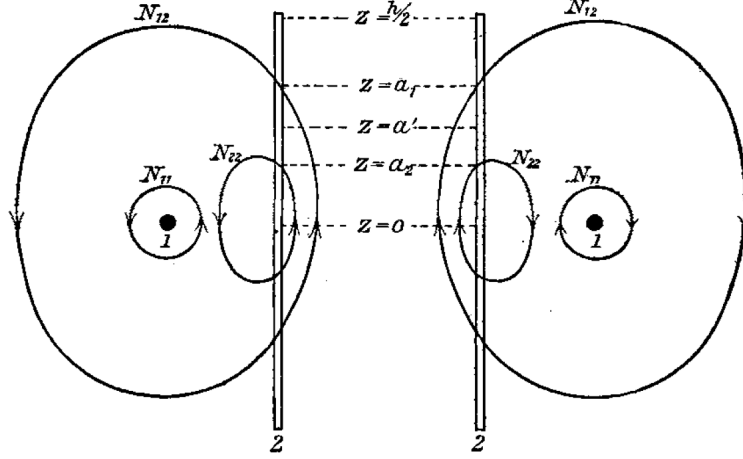


Figure 1: Reproduction of Fig. 1 on p. 515 of Drude [4]. Copyright Wiley-VCH Verlag GmbH & Co. KGaA. Reproduced with permission.

It is therefore, according to the laws of magnetic circuits, written:

$$N_{12} = \frac{4\pi i_1 + 8ni_2^0 \sin(\pi a_1/h)}{W_{12}}, \quad (6)$$

where  $W_{12}$  denotes the magnetic resistance of all the tube strengths  $N_{12}$  connected in parallel.

From (2) and (4), it follows:

$$4\pi \left( \frac{1}{W_{11}} + \frac{1}{W_{12}} \right) \frac{d^2 i_1}{dt^2} + \frac{8n}{W_{12}} \sin\left(\frac{\pi a_1}{h}\right) \frac{d^2 i_2^0}{dt^2} + w_1 \frac{di_1}{dt} + \frac{i_1}{C_1} = 0. \quad (7)$$

The differential equation for the current strength  $i_2$  in the coil includes  $t$  and  $z$  as independent variables.

If  $\epsilon_2$  denotes the electric charge (after electromagnetic measurements) along the horizontal length  $dz = 1$  of the coil,  $\mathfrak{C}_2$  the capacitance of this length, and  $V_2$  the potential of the coil at the position  $z$ , then it follows:

$$\frac{\partial i_2}{\partial z} = -\frac{\partial \epsilon_2}{\partial t}, \quad \epsilon_2 = \mathfrak{C}_2 V_2. \quad (8)$$

Moreover,  $N_2$  is the number of magnetic field lines that cross through the cross-section of the coil at height  $z$ , so it follows that between the positions  $z$  and  $z + dz$ , between which  $(n/h)dz$  windings lie, that  $i_2^2 \mathfrak{w}_2 dz$  is the energy dissipation per time (from resistance and radiation):

$$i_2 \mathfrak{w}_2 = -\frac{n}{h} \frac{\partial N_2}{\partial t} - \frac{\partial V_2}{\partial z}. \quad (9)$$

From (8) and (9) it follows

$$\frac{n}{h} \frac{\partial^2 N_2}{\partial t^2} + \mathfrak{w}_2 \frac{\partial i_2}{\partial t} - \frac{1}{\mathfrak{C}_2} \frac{\partial^2 i_2}{\partial z^2} = 0. \quad (10)$$

We can rewrite this partial differential equation completely in terms of  $i_2^0$ , if we also write  $N_2$  as a Fourier series, with the leading term remaining

$$N_2 = N_2^0 \cos\left(\frac{\pi z}{h}\right), \quad (11)$$

and if we set  $\mathfrak{w}_2$  and  $\mathfrak{C}_2$ , which depend simply on  $z$ , to their average across the whole coil; that is, we treat them as constants. Then (3), (10), and (11) yield

$$\frac{n}{h} \frac{\partial^2 N_2^0}{\partial t^2} + \mathfrak{w}_2 \frac{di_2^0}{dt} + \frac{\pi^2}{h^2 \mathfrak{C}_2} i_2^0 = 0. \quad (12)$$

From (11),  $N_2^0$  signifies the quantity of field lines that pass through the coil cross-section at the middle of the coil  $z = 0$ . This quantity can also be separated into two parts

$$N_2 = N_{22} + N_{12}, \quad (13)$$

where  $N_{22}$  are the field lines that only pass through the coil, but do not go around circuit 1, while  $N_{12}$  wind around the coil-windings as well as the primary circuit 1. For  $N_{22}$  the magnetic force strength is no longer constant; we can however guess for it an average magnetic force strength

$$8ni_2^0 \sin\left(\pi \frac{a_2}{h}\right)$$

where  $a_2 < a'$ , which is to say that  $a_2 < a_1$  also. From this it follows that

$$N_{22} = \frac{8ni_2^0 \sin\left(\pi \frac{a_2}{h}\right)}{W_{22}}, \quad (14)$$

if  $W_{22}$  is the magnetic resistance of the  $N_{22}$  tube strengths. In contrast, (12) yields:

$$8n^2 \left( \frac{\sin\left(\pi \frac{a_1}{h}\right)}{W_{12}} + \frac{\sin\left(\pi \frac{a_2}{h}\right)}{W_{22}} \right) \frac{d^2 i_2^0}{dt^2} + \frac{4\pi n}{W_{12}} \frac{d^2 i_1}{dt^2} + \mathfrak{w}_2 h \frac{di_2^0}{dt} + \frac{\pi^2}{\mathfrak{C}_2 h} i_2^0 = 0. \quad (15)$$

This equation in combination with equation (7) lays the framework for our problem. One can now write, for simplification,  $i_2$  for  $i_2^0$ , that is to say *from now on  $i_2$  means the current strength in the middle winding of the Tesla coil*, so we have our framework in the well-known form:

$$\begin{cases} L_{11} \frac{d^2 i_1}{dt^2} + L_{12} \frac{d^2 i_2}{dt^2} + w_1 \frac{di_1}{dt} + \frac{i_1}{C_1} = 0, \\ L_{22} \frac{d^2 i_2}{dt^2} + L_{21} \frac{d^2 i_1}{dt^2} + w_2 \frac{di_2}{dt} + \frac{i_2}{C_2} = 0. \end{cases} \quad (16)$$

From there it follows:

$$\begin{cases} L_{11} = 4\pi \left( \frac{1}{W_{11}} + \frac{1}{W_{12}} \right), & L_{22} = \frac{16n^2}{\pi} \left( \frac{\sin\left(\pi \frac{a_1}{h}\right)}{W_{11}} + \frac{\sin\left(\pi \frac{a_2}{h}\right)}{W_{22}} \right), \\ L_{12} = \frac{8n}{W_{12}} \sin\left(\pi \frac{a_1}{h}\right), & L_{21} = \frac{8n}{W_{12}}, \\ w_2 = \frac{2}{\pi} \mathfrak{w}_2 h, & C_2 = \mathfrak{C}_2 h : 2\pi. \end{cases} \quad (17)$$

From (17) it follows

$$L_{12} : L_{21} = \sin\left(\pi \frac{a_1}{h}\right), \quad (18)$$

that is, *we can no longer, as with two linear circuits, set  $L_{12} = L_{21}$ , but rather  $L_{12} < L_{21}$* , and even more so the smaller  $a_1$  is; that is to say, the nearer to the primary circuit that the middle winding of the Tesla coil lies and the higher the Tesla coil is in relation to the primary circuit. *At various positions of the Tesla coil relative to the primary circuit the relationship  $L_{12} : L_{21}$  alternates, and it becomes even smaller, the stronger the mutual induction (magnetic coupling) between the Tesla coil and the primary circuit becomes.* This result, whose derivation came from the interpretation of the coefficients of the well-known equations (16), *also holds for any, including asymmetrical, positions of the Tesla coil relative to the primary circuit.*

Of the coefficients  $L$ ,  $L_{11}$  and  $L_{21}$  are relatively easy to calculate theoretically, while  $L_{22}$  and  $L_{12}$  may only be obtained after very tedious calculation.  $L_{21}$  arises from the field lines, which current  $i_i$  sends through the cross-sectional area  $q$  of the Tesla coil. We call this count, if  $i_1 = 1$ , around any position  $z$  of the coil  $N_{21}$ , so the first term  $N_{21}^0$  of the Fourier series is

$$N_{21} = N_{21}^0 \cos\left(\frac{\pi z}{h}\right)$$

given through

$$N_{21}^0 = \frac{2}{h} \int_{-h/2}^{+h/2} N_{21} \cos\left(\frac{\pi z}{h}\right) dz. \quad (19)$$

Therefore,

$$L_{21} = \frac{4n}{\pi h} \int_{-h/2}^{+h/2} N_{21} \cos\left(\frac{\pi z}{h}\right) dz. \quad (20)$$

If the primary windings are circuits, then the magnetic force originated through  $i_1$  may be represented by spherical harmonics at any point in space, and from there  $N_{21}$ , and thus also  $L_{21}$ , can be calculated. For the self-induction  $L_{11}$ , known formulas have already been worked out.<sup>12</sup> One must not use this formula for

$L_{22}$ , since  $i_2$  is not constant along coil 2.

From equation (16), one can write equations for the potential difference  $V_1$  across the capacitor  $C_1$  in circuit 1, and for the potential  $V_2$  (actually written as  $V_2^h$ ) at a free end  $z = h/2$  of the coil. With (1) and (8), it can be worked out from (16) that:<sup>13</sup>

$$\begin{cases} L_{11}C_1 \frac{d^2 V_1}{dt^2} - 2L_{12}C_2 \frac{d^2 V_2}{dt^2} + w_1 C_1 \frac{dV_1}{dt} + V_1 = 0, \\ L_{22}C_2 \frac{d^2 V_2}{dt^2} - \frac{L_{21}C_1}{2} \frac{d^2 V_1}{dt^2} + w_2 C_2 \frac{dV_2}{dt} + V_2 = 0. \end{cases} \quad (21)$$

We will rather use these equations in the form:

<sup>12</sup>Stefan's formulas for the coils with oscillatory condenser discharges need a correction, see P. Drude., Ann. Phys. 9. p. 604. 1902.

<sup>13</sup>These equations take the form of the equations by M. Wien (Ann. d. Phys. 8. p. 694. 1902), if one reflects, that  $2V_2$  is the potential difference between both coil ends. From there follows the valid equation

$$i_2^0 = 2C_2 \frac{dV_2^h}{dt}.$$

The following developments are valid also for the case that the secondary circuit arises not from a coil, but from a linear circuit with a large added capacitor  $C_2$ .  $2V_2$  is then the secondary potential difference.

## Summary of Main Results

1. From the combined observation of the maximum-amplitude resonance curve and the integral effects one can find *the damping of both oscillating circuits uniquely, as well as the frequency of the oscillation.*

2. The resonance curve becomes more pronounced, the weaker the coupling between the oscillating circuits is. Further, it is more pronounced for the integral effects than for the maximum amplitude.

3. The most effective Tesla transformer comes from *one* primary coil and *many* secondary coils, which constitute a body of coils *with a particular* (not yet determined) *ratio of the height to the diameter.* The quantity of secondary coils is limited through the requirement that the insulation is not punctured and the wire thickness is not too thin. Further, an increase in the number of secondary coils requires a spark-inductor of higher sparking length. Dead (not inductively effective) self-induction of the primary coil is possible to work around; its wire thickness should be as large as possible and *the coupling between the primary coil and the Tesla coil should be as close as possible to the value  $k^2 = 0.36$*  (ratio of the frequencies of 1:2, originating through the coupling). The primary capacitor must bring the primary circuit in resonance with the Tesla

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coil and should (to protect from electric discharge and residue) be in well-insulated, residue-free dielectric (oil, not glass or air), out of metal contact.

With weak coupling, it matters only *that the primary capacitance  $C_1$  is as large as possible*, irrespective of whether the couplings are achieved through small coils with high winding number  $n$ , or through large coils with smaller  $n$ ; with strong coupling, high  $n$  is somewhat more efficient. — Within certain limits, the tesla coil depends little on sparking potential.

4. With weak coupling, the damping of the primary and secondary circuits has a strong influence on the effectiveness of the Tesla transformers; with stronger coupling (even from  $k^2 = 0.16$  on) much less.

5. With radio telegraphy one finds the sharpest resonance (neglecting the intensity) through weakly coupled and undamped sender and receiver. The latter should include a meter that responds to the integral effect.

6. If one has a weakly coupled (or separated) receiver and a strongly coupled sender, there is no distinct resonance. *First through much stronger coupling in the sender ( $k^2 > 0.6$ ) can one tune the receiver to the sender. The ratios of the receiver frequency to the frequencies of both (coordinated) oscillating circuits of the sender must then be less than  $1 : \sqrt{2}$ .*

7. If one has two identically built and identically coupled instruments as sender and receiver, then one can achieve a high intensity and moderate precision in the resonance, the latter more from the integral effect than from the maximum amplitude. To achieve the best performance, the coupling in both instruments should be  $k^2 = 0.36$ . By using a meter sensitive to the maximal amplitude, the outcome depends less on the damping, so that the resonance becomes less precise than if one had measured it with the integral effect.

Giessen, November 1903.

(Received 25 November, 1903.)

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End of article and translation

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## 4 Discussion

In the beginning of the article, Drude analyzes a half-wave (or bipolar) Tesla transformer operating near the fundamental (uncoupled) self-resonance frequency of its secondary circuit. The primary circuit is treated as an ideal lumped-element circuit. The secondary circuit is a single-layer solenoid without any capacitive loading (no discharge terminals) or ground connection.

Drude's approach is essentially transmission-line analysis, although he treats inductance using Hopkinson's law, which is described in Sec. 4.2. For example, (8) and (9) are the Telegrapher's equations for a transmission line with constant, distributed shunt capacitance  $\mathfrak{C}_2$  and series resistance  $\mathfrak{w}_2$ . He expands  $i_2(z)$  and  $V_2(z)$  into Fourier series for spatial standing waves along the secondary coil, and retains only the fundamental spatial mode. In the end, he derives a system of equations which represents an equivalent circuit for a half-wave Tesla transformer.

In this section, we provide a modern interpretation of this analysis, and extend the results to a quarter-wave Tesla transformer. We list the errors we noticed in the article, and also describe possible modifications to match the equivalent secondary self-inductance with that of previous work by Drude and others [2, 17].

### 4.1 Technical glossary

Here is a modern interpretation of some of the language in the translation:

**maximum amplitude, integral effect** (first uses p. 512) two wireless telegraphy detection techniques.

**current strength** (first use p. 513 with  $i_1$ ) current.

**damping coefficient** (first use p. 514 with  $w_1$ ) resistance.

**magnetic field lines** (first use p. 514 with  $N_1$ ) magnetic flux.

**magnetomotive strength** (first use p. 514 with  $4\pi i_1$ ) same as magnetic force (see below).

**magnetic force** (first use p. 515) magnetomotive force (MMF).

**magnetic resistance** (first use pp. 514–5 with  $W_{11}$ ) reluctance.

**law of magnetic circuits** (first use p. 516) Hopkinson's law.

**tube strengths** (first use p. 516 with  $N_{12}$ ) same as magnetic field lines (see above).

### 4.2 Hopkinson's law

Hopkinson's law (or Rowland's law) for magnetic circuits resembles Ohm's law for electric circuits [18]. In Ohm's law, the electromagnetic force (EMF, or voltage) across some element is equal to the current  $I$  passing through it times its electrical resistance  $R$ :

$$\text{Ohm's law: } \text{EMF} = IR. \quad (22)$$

In Hopkinson's law, the magnetomotive force (MMF) across some element is equal to the magnetic flux  $\Phi_m$  through it times its reluctance  $R_m$ :

$$\text{Hopkinson's law: } \text{MMF} = \Phi_m R_m. \quad (23)$$

Comparing both laws, we see that the MMF plays the role of an EMF, the magnetic flux  $\Phi_m$  of a current  $I$ , and the reluctance  $R_m$  of an electrical resistance  $R$ .

As an example, consider an ideal lumped-element inductor with  $N$  turns (not to be confused with Drude's  $N$ ), self-inductance  $L$ , and current  $I$  flowing through it. The MMF given by Ampere's law is  $NI$ , and can be thought of as the equivalent current if there was only a single turn ( $N = 1$ ). The magnetic flux  $\Phi_m$  is almost the same as the flux in the definition of self-inductance ( $\Phi = LI$ ), except that for magnetic circuits it is typically the flux per turn,  $\Phi_m = LI/N$ . From Hopkinson's law, the reluctance is  $R_m = \text{MMF}/\Phi_m = N^2/L$ . For a single-turn inductor ( $N = 1$ ), the reluctance is just the reciprocal of the inductance.

In the translation, Eq. (5) shows that Drude's MMF has an extra factor of  $4\pi$ , likely from using cgs units (with  $c = 1$ ). Eq. (14) shows that his MMF is proportional to the number of turns, following Ampere's law.

Eq. (9) suggests that his magnetic flux is a flux per turn, following  $\Phi_m$ . As a result, his reluctance should be of the form  $4\pi R_m$ .

### 4.3 Equivalent circuit parameters from the article

Systems (16) and (21) describes an equivalent circuit for an unloaded, half-wave Tesla transformer. With some changes, we can relate this circuit to the conventional equivalent circuit for a Tesla transformer, which is described by the system

$$I_p(t) + C_p \frac{dV_p(t)}{dt} = 0 \quad (24)$$

$$L_p \frac{d^2 I_p(t)}{dt^2} + R_p \frac{dI_p(t)}{dt} + \frac{1}{C_p} I_p(t) + M_{ps} \frac{d^2 I_s(t)}{dt^2} = 0 \quad (25)$$

$$I_s(t) + C_s \frac{dV_s(t)}{dt} = 0 \quad (26)$$

$$L_s \frac{d^2 I_s(t)}{dt^2} + R_s \frac{dI_s(t)}{dt} + \frac{1}{C_s} I_s(t) + M_{sp} \frac{d^2 I_p(t)}{dt^2} = 0. \quad (27)$$

Here, the first two lines describing the primary circuit follow directly from (1) and (16). The primary circuit current  $I_p(t) = i_1$ , voltage  $V_p(t) = V_1$ , inductance  $L_p = L_{11}$ , capacitance  $C_p = C_1$ , and resistance  $R_p = w_1$ .

However, the third and fourth lines of (27) require some care with factors of 2. Following Drude, the secondary voltage  $V_s(t) = V_2$  of (21) is the potential difference between one free end to the middle winding ( $V_2^h$ ), and the secondary current  $I_s(t) = i_2$  of (16) is the middle winding's current ( $i_2^0$ ). The third line comes from footnote 13 on p. 519, which gives the capacitance  $C_s = 2C_2$ . The fourth line then follows from dividing (16) by two, or (21) by  $C_s$ , which gives the inductance  $L_s = L_{22}/2$  and resistance  $R_s = w_2/2$ .

Note, however, that Drude predicts a nonreciprocal mutual inductance,  $M_{ps} \neq M_{sp}$ . From (16) and (17), the effective mutual inductance seen by the primary is  $M_{ps} = L_{12}$ . Likewise, for the half-wave case treated by Drude, the effective mutual inductance seen by the secondary is  $M_{sp} = L_{21}/2$ . Unfortunately, the coefficients  $L_s$  and  $M_{ps}$  are left in terms of the unknown parameters  $\sin(\pi a_1/h)$  and  $\sin(\pi a_2/h)$ .

To extend the results to a quarter-wave Tesla transformer, consider removing the  $z < 0$  region of the secondary coil. Conveniently, the secondary voltage and current definitions still hold and, with the exception of the mutual inductances, most of the circuit parameters are unchanged. We will return to this in Sec. 4.5.

Finally, note that the above relations follow directly from the translation, and do not account for the errors mentioned in the next section.

### 4.4 Errors in the article

The equations in this translation copy the article. Errors we noticed are listed here:

- pp. 516-7:  $N_2$  was used instead of the Fourier term  $N_2^0$  in (12) to get (15).
- p. 517: The  $W_{11}$  in the expression for  $L_{22}$  in (17) does not match the  $W_{12}$  in (15), and probably should be a  $W_{12}$ .
- p. 518: After (18), the discussion of  $N_{21}$  ignores the contribution from  $i_2$ . This seems to be inconsistent with the earlier interpretation of  $N_{12}$  and  $N_{21}$ . For example, (13) implies  $N_{12} = N_{21}$ , and (6) gives  $N_{12}$  with contributions from both  $i_1$  and  $i_2$ .
- p. 518: The expression for  $L_{21}$  in (20) seems to be missing a factor of  $1/W_{12}$ .

### 4.5 Modifications to match $L_s$ with previous work

The article gives the equivalent secondary coil self-inductance  $L_s = L_{22}/2$  in terms of the unknown parameters  $\sin(\pi a_1/h)$  and  $\sin(\pi a_2/h)$  in (17). This result differs from previous work by Drude (and others) for quarter-

wave Tesla transformers, which give the result

$$L_s = \left(\frac{2}{\pi}\right) lH, \quad (28)$$

where  $H = h/2$  is the height of the secondary coil and  $l$  is a distributed series inductance, such that  $lH = L_s^{(\text{dc})}$  is the low-frequency self-inductance of the coil [2, 17]. (However, modern equivalent circuits for solenoids typically use the this low-frequency self inductance instead of (28) [17].) Interestingly, the two parameters in (17) that are expressed in terms of distributed quantities share a similar form,

$$C_s = 2C_2 = \left(\frac{2}{\pi}\right) \mathfrak{C}_2 H \quad \text{and} \quad R_s = \frac{w_2}{2} = \left(\frac{2}{\pi}\right) \mathfrak{w}_2 H. \quad (29)$$

The similarity of (28) with (29) is suspicious, and raises the question of why Drude did not obtain the same result (28) again for  $L_s$ . One possible explanation is that Drude's use of magnetic fluxes  $N$  with Fourier series is incorrect. (It is at least inconsistent.) What follows is a modified derivation which recovers the previous result (28).

First, note that there was an error going from (12) to (15):  $N_2$  was used for (15) in place of the Fourier term  $N_2^0$  in (12). To fix this, we need to calculate the  $N_2^0$  of (11). We can suppose that  $N_2^0 = N_{22}^0 + N_{21}^0$ ,

$$N_{22}^0 = \frac{4\pi}{W_{22}} \left(\frac{2}{h}\right) \int_{-h/2}^{h/2} n i_2(z) \cos\left(\frac{\pi z}{h}\right) dz = \frac{4\pi n i_2^0}{W_{22}}, \quad (30)$$

$$\text{and that } N_{21}^0 = \frac{4\pi}{W_{21}} \left(\frac{2}{h}\right) \int_{-h/2}^{h/2} [i_1 + n i_2(z)] \cos\left(\frac{\pi z}{h}\right) dz = \frac{16i_1 + 4\pi n i_2^0}{W_{21}}. \quad (31)$$

(Note that the  $N_{21}^0$  on p. 518 is a different quantity.) As a result, in (15) we should have the substitutions

$$8n^2 \left( \frac{\sin(\pi a_1/h)}{W_{21}} + \frac{\sin(\pi a_2/h)}{W_{22}} \right) \longrightarrow 4\pi n^2 \left( \frac{1}{W_{21}} + \frac{1}{W_{22}} \right) \quad \text{and} \quad \frac{4\pi n}{W_{12}} \longrightarrow \frac{16n}{W_{21}}. \quad (32)$$

The first substitution changes the  $L_{22}$  in (17) to

$$L_{22} = \left(\frac{2}{\pi}\right) 4\pi n^2 \left( \frac{1}{W_{21}} + \frac{1}{W_{22}} \right). \quad (33)$$

However, to treat a quarter-wave Tesla transformer, (30) and (31) should be changed as follows: the total number of turns  $n$  should be reduced to  $n' = n/2$ , the integrals should be restricted to  $[0, H]$ , where  $H = h/2$ , and the pre-factors  $2/h$  replaced with  $2/H$ . With these changes, (30) and (31) become

$$N_{22}^0 = \frac{4\pi}{W_{22}} \left(\frac{2}{H}\right) \int_0^H n' i_2(z) \cos\left(\frac{\pi z}{h}\right) dz = \frac{4\pi n' i_2^0}{W_{22}} \quad (34)$$

$$\text{and } N_{21}^0 = \frac{4\pi}{W_{21}} \left(\frac{2}{H}\right) \int_0^H [i_1 + n' i_2(z)] \cos\left(\frac{\pi z}{h}\right) dz = \frac{16i_1 + 4\pi n' i_2^0}{W_{21}}. \quad (35)$$

Comparing (30-31) with (34-35), we find from (33) that the  $L_s$  for a quarter-wave Tesla transformer is

$$L_s = \frac{L_{22}}{2} = \left(\frac{2}{\pi}\right) 4\pi (n')^2 \left( \frac{1}{W_{21}} + \frac{1}{W_{22}} \right). \quad (36)$$

Finally, inspecting the form of  $L_p = L_{11}$  in (17), we can suppose that  $lH = L_s^{(\text{dc})} = 4\pi(1/W_{21} + 1/W_{22})$ , so we see that the result (36) is of the desired form (28).

These modifications also change the results for mutual inductance. For the half-wave case, the second substitution in (32) changes the  $L_{21}$  in (17) to

$$L_{21} = \left(\frac{2}{\pi}\right) \frac{16n}{W_{21}}. \quad (37)$$

Note that this also holds for the quarter-wave case, since the factor of  $n$  here comes from the turn density  $n/h = N/H$  introduced in (12). Therefore, the mutual inductance for both cases is

$$M_{\text{sp}} = \frac{L_{21}}{2} = \left(\frac{2}{\pi}\right) \frac{8n}{W_{21}} = \left(\frac{4}{\pi}\right) \frac{8n'}{W_{21}}. \quad (38)$$

Focusing now on the primary, in place of the derivation of  $N_{12}$  on p. 515, we can suppose that

$$N_{12} = \frac{4\pi}{W_{12}} \left[ i_1 + \frac{n}{h} \int_{-h/2}^{h/2} i_2(z) dz \right] = \frac{4\pi i_1 + 8n i_2^0}{W_{12}} \quad (39)$$

for the half-wave case. As a result, in (7) we should have the substitution

$$\frac{8n}{W_{12}} \sin\left(\frac{\pi a_1}{h}\right) \rightarrow \frac{8n}{W_{12}}, \quad (40)$$

which changes the  $L_{12}$  in (17) to

$$L_{12} = \frac{8n}{W_{12}}. \quad (41)$$

However, following the changes described above, for the quarter-wave case we should have

$$N_{12} = \frac{4\pi}{W_{12}} \left[ i_1 + \frac{n'}{H} \int_0^H i_2(z) dz \right] = \frac{4\pi i_1 + 8n' i_2^0}{W_{12}}, \quad (42)$$

which again leads to the same result (41) if  $n$  is replaced with  $n'$ . Therefore, the mutual inductance

$$M_{\text{ps}} = L_{12} = \frac{8n}{W_{21}} \text{ for the half-wave case, or } \frac{8n'}{W_{21}} \text{ for the quarter-wave case.} \quad (43)$$

Using (43) with (38), we find that the ratio of the mutual inductances

$$\frac{M_{\text{ps}}}{M_{\text{sp}}} = \frac{2L_{12}}{L_{21}} = \frac{\pi}{2} \text{ for the half-wave case, and } \frac{\pi}{4} \text{ for the quarter-wave case.} \quad (44)$$

We see that Drude's prediction on p. 518 that  $L_{12} < L_{21}$  still holds, though the unknown parameter  $\sin(\pi a_1/h)$  has been eliminated.

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